Degenerate conic decomposition

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Definition 1. *We define a line as the set of points $(x,\ y)$ in $\mathbb{R}^2$ that satisfies the equation $a\,x + b\,y + c = 0$ where at least one between a and b is not zero.*

Definition 2. *We define a conic as the set of points $(x,\ y)$ in $\mathbb{R}^2$ that satisfies the equation $a\,x^2 + b\,x\,y + c\,y^2 + d\,x + e\,y + f = 0$ where at least one between a, b and c is not zero.*

Definition 3. *Given a conic C we define the symmetric matrix related to C as*

$$M_C = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}.$$

Set $P = (x, y, 1)$ the conic equation can be written as $P^T M_C\, P = 0$.

Definition 4. *We say that a conic C is a degenerate conic when can be decomposed as the union of two lines.*

Given a degenerate conic C let be r_1 and r_2 the pair of lines that C is made of. Let be r_1 defined by the equation $a_1x + b_1y + c_1 = 0$ and r_2 defined by the equation $a_2x + b_2y + c_2 = 0$. We can define the matrix

$$L = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 & a_1c_2 \\ b_1a_2 & b_1b_2 & b_1c_2 \\ c_1a_2 & c_1b_2 & c_1c_2 \end{pmatrix}.$$

Remark 1. Every non-null row of L still defines the line r_2; every non-null column of L still defines the line r_1.

Definition 5. *Given a square matrix M we call $\mathcal{S}(M) = (M + M^T)/2$ and $\mathcal{A}(M) = (M - M^T)/2$ the symmetric and antisymmetric parts of M.*

Lemma 1. *Given a square matrix M we have $M = \mathcal{S}(M) + \mathcal{A}(M)$.*

The symmetric matrix M_C related to a degenerate conic C is the symmetric part of L:

$$M_C = \mathcal{S}(L) = \frac{1}{2}\begin{pmatrix} 2a_1a_2 & a_1b_2 + b_1a_2 & a_1c_2 + c_1a_2 \\ a_1b_2 + b_1a_2 & 2b_1b_2 & b_1c_2 + c_1b_2 \\ a_1c_2 + c_1a_2 & b_1c_2 + c_1b_2 & 2c_1c_2 \end{pmatrix}.$$

When r_1 and r_2 are the same line we have $a_2 = \lambda\, a_1$, $b_2 = \lambda\, b_1$, $c_2 = \lambda\, c_1$ for some $\lambda \neq 0$. So the symmetric matrix M_C becomes:

$$\lambda \begin{pmatrix} a_1^2 & a_1b_1 & a_1c_1 \\ a_1b_1 & b_1^2 & b_1c_1 \\ a_1c_1 & b_1c_1 & c_1^2 \end{pmatrix}.$$

Remark 2. When C is a double line any non-null row and any non-null column of M_C defines the line $r_1 \equiv r_2$.

Definition 6. *Given a square matrix A with order n we define $A_{i,j}$ as the square matrix with order $n-1$ obtained by deleting the i-th row and the j-th column of A.*

Definition 7. *Given a square matrix A we define the adjoint of A as the matrix $D = \mathrm{adj}(A)$ such that $d_{i,j} = (-1)^{i+j} \det(A_{j,i})$.*

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When C is degenerate the adjoint of M_C is

$$\mathrm{adj}(M_C)=\begin{pmatrix} -(b_1c_2-c_1b_2)^2 & (a_1c_2-c_1a_2)(b_1c_2-c_1b_2) & -(a_1b_2-b_1a_2)(b_1c_2-c_1b_2) \\ (a_1c_2-c_1a_2)(b_1c_2-c_1b_2) & -(a_1c_2-c_1a_2)^2 & (a_1b_2-b_1a_2)(a_1c_2-c_1a_2) \\ -(a_1b_2-b_1a_2)(b_1c_2-c_1b_2) & (a_1b_2-b_1a_2)(a_1c_2-c_1a_2) & -(a_1b_2-b_1a_2)^2 \end{pmatrix}$$

Proposition 1. *If C is a degenerate conic then C is made up by two different lines r_1, r_2 iff* $\mathrm{rank}(M_C)=2$*; C is a double line iff* $\mathrm{rank}(M_C)=1$.

Proof. When $\mathrm{rank}(M_C)<2$ then $\mathrm{adj}(M_C)$ is the null matrix, so we have

$$\begin{aligned} a_1c_2-c_1a_2=0,\\ a_1b_2-b_1a_2=0 \end{aligned}$$

Since we suppose that r_1 and r_2 are really two lines we can think that at least one between a_1 and b_1 is not zero, so let be $a_1\neq 0$. In the same way we can suppose that a_2 or b_2 is not zero.

If $a_2\neq 0$ we can divide $a_1c_2-c_1a_2=0$ and $a_1b_2-b_1a_2=0$ by $a_1\,a_2$ and we get:

$$\frac{c_1}{a_1}=\frac{c_2}{a_2},\quad \frac{b_1}{a_1}=\frac{b_2}{a_2}.$$

From this follows that r_1 and r_2 are the same line.

In case $a_2=0$ and $b_2\neq 0$ we have $a_1b_2-b_1a_2=0\Rightarrow a_1b_2=0$ but this is impossible because we have supposed that a_1 and b_2 are not zero.

Now let be r_1 and r_2 the same line so we can suppose that exists $\lambda\neq 0$ such that

$$a_2=\lambda\,a_1, b_2=\lambda\,b_1, c_2=\lambda\,c_1.$$

By substituting these values for a_2, b_2, c_2 in the above expression for $\mathrm{adj}(M_C)$ we can see immediately that we get the null matrix.

To end the demonstration is needed only to prove that when r_1 and r_2 are the same line then $\mathrm{rank}(M_C)>0$. This follows immediately looking at the diagonal elements in the expression for M_C, they are: $\lambda\,a_1^2$, $\lambda\,b_1^2$, $\lambda\,c_1^2$ as one between a_1 and b_1 is not zero the rank of the matrix is not zero too.

□

Proposition 2. *If C is a degenerate conic made up by two lines r_1, r_2, with $r_1\neq r_2$ then* $\mathrm{adj}(M_C)$ *is the matrix related to a degenerate conic too and exactly to a double line. When $r_1\equiv r_2$, i.e. C is a double line, we have that* $\mathrm{adj}(M_C)$ *is the null matrix.*

Proof. The last assertion follows from the previous proposition.

Now set $\alpha=\frac{1}{\sqrt{2}}(b_1c_2-c_1b_2)$, $\beta=\frac{1}{\sqrt{2}}(a_1c_2-c_1a_2)$, $\gamma=\frac{1}{\sqrt{2}}(a_1b_2-b_1a_2)$ we have:

$$\mathrm{adj}(M_C)=2\begin{pmatrix} -\alpha^2 & \alpha\beta & -\alpha\gamma \\ \alpha\beta & -\beta^2 & \beta\gamma \\ -\alpha\gamma & \beta\gamma & -\gamma^2 \end{pmatrix}=\mathcal{S}\left(\begin{pmatrix} \alpha \\ -\beta \\ \gamma \end{pmatrix}\begin{pmatrix} -\alpha & \beta & \gamma \end{pmatrix}\right).$$

Clearly, $\alpha\,x-\beta\,y+\gamma=0$ and $-\alpha\,x+\beta\,y-\gamma=0$ define the same line.

□

Corollary 1. *If C is a degenerate conic then* $\mathrm{rank}(\mathrm{adj}(M_C))=\mathrm{rank}(M_C)-1$.

Lemma 2. *Given a vector $v\in\mathbb{R}^3$, there is a linear application A: $\mathbb{R}^3\to\mathbb{R}^3$ such that $A(u)=v\times u\ \forall u\in\mathbb{R}^3$.*

Remark 3. If $v=(x,y,z)$ the matrix related to the linear application A is the following:

$$M_A=\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Note that this is an antisymmetric matrix. We call M_A the cross product matrix related to the vector v and we denote it by $\mathcal{CP}(v)$.

Proposition 3. *Let be C a degenerate conic made up by two lines r_1, r_2, with $r_1 \neq r_2$, let be p the vector $(\alpha, -\beta, \gamma)$ where $\alpha = b_1c_2 - c_1b_2$, $\beta = a_1c_2 - c_1a_2$, $\gamma = a_1b_2 - b_1a_2$. Then the cross product matrix related to the vector p is, up to a multiplicative factor, the antisymmetric part of $L = (a_1, b_1, c_1)^T (a_2, b_2, c_2)$. Moreover we can recover α, β, γ from* $\mathrm{adj}(M_C)$.

Proof. The first statement is trivial:

$$\mathcal{A}(L) = \frac{L - L^T}{2} = \frac{1}{2}\begin{pmatrix} 0 & a_1b_2 - b_1a_2 & a_1c_2 - c_1a_2 \\ -a_1b_2 + b_1a_2 & 0 & b_1c_2 - c_1b_2 \\ -a_1c_2 + c_1a_2 & -b_1c_2 + c_1b_2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \gamma & \beta \\ -\gamma & 0 & \alpha \\ -\beta & -\alpha & 0 \end{pmatrix} = \frac{1}{2}\,\mathcal{CP}(p).$$

Now, we start by noting that $r_1 \neq r_2$ implies $\mathrm{rank}(M_C) = 2$, hence at least one between α, β, γ is not zero, on the contrary $D = \mathrm{adj}(M_C)$ would be the null matrix. So without loss of genericity we can suppose $\alpha \neq 0$. This means that set $\lambda = \sqrt{-d_{1,1}}$ we have $\alpha = \lambda$, $\beta = \frac{d_{1,2}}{\lambda}$, $\gamma = -\frac{d_{1,3}}{\lambda}$.

□

Proposition 4. *Let be M_C the symmetric matrix related to a degenerate conic C made up by two lines r_1, r_2. We can recover the equation for r_1 and r_2 from the matrix M_C.*

Proof. First we prove the case $r_1 \equiv r_2$, i.e. C is a double line. Since $\mathrm{rank}(M_C) = 1$ there is a non-null element of M_C, say $m_{i,j}$ for some pair of indices (i, j). By remark 2 we can recover the equation of the line by choosing as coefficients the elements of the i-th row of M_C or the elements of the j-th column of M_C.

Now we prove the case $r_1 \neq r_2$. If $L = (a_1, b_1, c_1)^T (a_2, b_2, c_2)$ we have $M_C = \mathcal{S}(L)$ and we can recover the antisymmetric part $\mathcal{A}(L)$ from M_C by the previous proposition. So from M_C we can recover $L = \mathcal{S}(L) + \mathcal{A}(L)$. Found a non-null element $l_{i,j}$ of L, by remark 1 we have that the j-th column provides the equation of the line r_1 and the i-th row provides the equation of the line r_2. The existence of a non-null element of L is assured because of supposing r_1 and r_2 two actual lines.

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